Uzbek Mathematical Journal, 1996, No.4

About the asymptotic formula for spectral function of the Laplace-Beltrami operator on sphere

Anvarjon Akhmedov
Tashkent State University,
Department of Mathematical Prysics,
Vuzgorodok, Tashkent, Uzbekistan.
anv.akhmed@gmail.com

Abstract

In this work we established asymptotical behavior for Riesz means of the spectral function of the Laplace operator on unit sphere.

Keywords: *Riesz means, Cesaro means, spectral function.*
2000 Mathematics Subject Classification: *46F12, 35P20,42B05,42B08, 42B25*

1 Introduction

Let S^N is the N-dimensional unit sphere in R^{N+1}. The Laplace operator Δ:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_N^2}, u \in C_0^\infty(S^N)$$

in spherical coordinates in N dimensions, with the parametrization $x = r\theta \in R^{N+1}$ with $r \in [0, +\infty)$ and $\theta \in S^N$,

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{N}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_S f$$

where Δ_S is the Laplace-Beltrami operator on the N- dimensional sphere, or spherical Laplacian. One can also write the term

$$\frac{\partial^2 f}{\partial r^2} + \frac{N}{r}\frac{\partial f}{\partial r}$$

equivalently as

$$\frac{1}{r^N}\frac{\partial}{\partial r}\Big(r^N \frac{\partial f}{\partial r}\Big).$$

As a consequence, the spherical Laplacian of a function defined on $S^N \subset R^{N+1}$ can be computed as the ordinary Laplacian of the function extended to $R^{N+1} \setminus \{0\}$ so that it is constant along rays.

Let us denote by $\lambda_0, \lambda_1, ...$ the distinct eigenvalues of $-\Delta_S$, arranged in increasing order. Let H_k denote the eigenspace corresponding to λ_k. We call elements of H_k spherical harmonics of degree k. It is well known (see [17]) that $dim H_k = a_k$:

$$a_k = \begin{cases} 1, & if \quad k = 0, \\ N, & if \quad k = 1, \\ \dfrac{(N+k)!}{N!k!} - \dfrac{(N+k-2)!}{N!(k-2)!}, & if \quad k \geq 2 \end{cases} \tag{1}$$

For a_k we have $a_k \approx k^{N-1}$ as $k \to \infty$.

Let $\hat{A}$ is a self-adjoint extension of the Laplace operator Δ_S in $L_2(S^N)$ and if E_λ is the corresponding spectral resolution, then for all functions $f \in L_2(S^N)$ we have

$$\hat{A} f = \int_0^\infty \lambda d E_\lambda f.$$

The operator$\hat{A}$ has in$L_2(S^N)$ a complete orthonormal system of eigenfunctions

$$\{Y_1^{(k)}(x), Y_2^{(k)}(x), ..., Y_{a_k}^{(k)}(x)\} \subset H_k, k = 0, 1, 2, ...,$$

corresponding to the eigenvalues $\{l_k = k(k+N-1)\}, k = 0, 1, 2, ...$

It is easy to check that the operators E_λ have the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} Y_n(f, x), \tag{2}$$

where

$$Y_k(f, x) = \sum_{j=1}^{a_k} Y_j^{(k)}(x) \int_{S^N} f(y) Y_j^{(k)}(y) d\sigma(y) \tag{3}$$

.

The Riesz means of the partial sums (2) is defined by

$$E_n^\alpha f(x) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha Y_k(f, x) \tag{4}$$

The most convenient object for a detailed investigation are the expansions of the form (4). The integral (4) may be transformed writing instead of $\hat{A}$ the integral to the right in (3) and then changing the order of integration. This yields the formula

$$E_\lambda^s f(x) = \int_{S^N} \Theta^s(x, y, \lambda) f(y) d\sigma(y) \tag{5}$$

with

$$\Theta^{\alpha}(x,y,n)=\sum_{k=0}^{n}\left(1-\frac{\lambda_k}{\lambda_n}\right)^{\alpha} Z_k(x,y). \tag{6}$$

For $\alpha=0$ this kernel is called the spectral function of the Laplace operator for the entire space S^N.

The behavior of the spectral expansion corresponding to the the Laplace-Beltrami operator is closely connected with the asymptotical behavior of the kernel $\Theta^{\alpha}(x,y,n)$.

For any two points x and y from S^N we shall denote by $\gamma(x,y)$ spherical distance between these two points. Actually, $\gamma(x,y)$ is a measure of angle between x and y. It is obvious, that $\gamma(x,y)\leq\pi$.

We proceed to the formulation of the fundamental results of the paper.

Theorem 1.1 *Let $\Theta^{\alpha}(x,y,n)$ is the kernel of Riesz means of the spectral expansions*
1)if $|\frac{\pi}{2}-\gamma|<\frac{n}{n+1}\frac{\pi}{2}$ then we have

$$\Theta^{\alpha}(x,y,n)=O(1)\Big(\frac{n^{(N-1)/2-\alpha}}{(\sin\gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}}+\frac{n^{(N-3)/2-\alpha}}{(\sin\gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}}+$$

$$+\frac{n^{-1}}{(\sin(\gamma)/2)^{1+N}}\Big);$$

2) if $0\leq\gamma\leq\pi$, then we have

$$\Theta^{\alpha}(x,y,n)=O(1)n^{N};$$

3) if $0<\gamma_0\leq\gamma\leq\pi$, then we have

$$\Theta^{\alpha}(x,y,n)=O(1)n^{N-\alpha};$$

2 Preliminaries

In this section we give some properties of Riesz means of spectral expansions.

Let f be a function with the support in $(0,+\infty)$. If f has the local bounded variation, then the Riesz means f^{α} for all $Re(\alpha)>-1$ is defined by

$$f^{\alpha}(t)=\int_0^t\left(1-\frac{s}{t}\right)^{\alpha} df(s). \tag{7}$$

Using the properties of the distribution $R_{\alpha}(t)=\alpha t^{\alpha-1}$, in [8] was established the generalization of M.Riesz theorem :

Theorem 2.1 *Let ζ is complex number with $Re(\zeta) > 0$, and $M_0(t)$ and $M_1(t)$ are positive increasing functions in $(0, +\infty)$. If $f(t) = 0, t < 0$, and for all $t > 0$ satisfied inequalities:*

$$|f(t)| \leq M_0(t) \tag{8}$$

$$|t^\zeta f^\zeta(t)| \leq M_1(t) \tag{9}$$

then for $0 < Re(\alpha) < Re(\zeta)$ we have

$$|t^\alpha f^\alpha(t)| \leq C(1+|\alpha|)^{Re(\zeta)+2} \left(\frac{|\alpha|}{Re(\alpha)} + \frac{|\zeta - \alpha|}{Re(\zeta - \alpha)} \right) M_0^{\frac{Re(\zeta-\alpha)}{Re(\zeta)}}(t) M_1^{\frac{Re(\alpha)}{Re(\zeta)}}. \tag{10}$$

Constant C depends only on ζ.

We are going to prove the Theorem 15 using this statement on Riesz means.

Let us introduce the Cesaro means of spectral expansions. The Cesaro means of order $\alpha \geq 0$ is defined as

$$C_n^\alpha f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} Y_k(f, x) \tag{11}$$

where $Y_k(f, x) \in H_k, k \geq 0$ and $A_m^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}, m = 0, 1, 2, ..., \Gamma(z)$ is Gamma function.

As a Riesz means the Cesaro means are integral operator with the kernel

$$\Xi^\alpha(x, y, n) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} Z_k(x, y), \tag{12}$$

where Z_k is zonal harmonics of order k, which is reproducing kernel for the space $H_k, k \geq 0$. For the kernel we have

Lemma 2.2 *If $\alpha > -1$ and $|\frac{\pi}{2} - \gamma(x, y)| \leq \frac{n}{n+1}\frac{\pi}{2}$, then*

$$\frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma(n+\frac{N+1}{2})}{\Gamma(\frac{N+1}{2})} \frac{\sin\left((n+\frac{N+1}{2})\gamma - (\frac{N-1}{2}+\frac{\alpha}{2})\frac{\pi}{2}\right)}{(2\sin\gamma)^{(N-1)/2}(2\sin(\gamma/2))^{1+\alpha}} +$$

$$+ \frac{O\left(n^{(N-3)/2}\right)}{(\sin\gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{O\left(n^{-1}\right)}{(\sin(\gamma)/2)^{1+N}});$$

If $\alpha > -1$ and $0 < \gamma_0 \leq \gamma \leq \pi$, then for all $n > 1$

$$|\Xi^\alpha(x, y, n)| \leq cn^{N-1-\alpha},$$

If $\alpha > -1$ and $0 < \gamma_0 \leq \gamma \leq \pi$, then for all $n > 1$

$$|\Xi^\alpha(x, y, n)| \leq cn^N.$$

This lemma is proved in [9]. It is well known that for integer order α kernels $\Theta^\alpha(x,y,n)$ and $\Xi^\alpha(x,y,n)$ have same asymptotical behavior. Our purpose to extend this result for all α.

Let us denote by

$$M_\alpha(t)=\frac{t^{(N-1)/2}}{(\sin\gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}}+\frac{t^{(N-3)/2}}{(\sin\gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}}+\frac{t^{-1}}{(\sin(\gamma)/2)^{1+N}};$$

If ζ is integer, then for the kernel $\Theta^\zeta(x,y,n)$ we have

$$|\Theta^0(x,y,n)|\le M_0(t);$$

and

$$t^\zeta|\Theta^\zeta(x,y,n)|\le M_\zeta(t).$$

So using the statement of Theorem 2.1 we obtain

$$|t^\alpha\Theta^\alpha(t)|\le C(1+|\alpha|)^{Re(\zeta)+2}\left(\frac{|\alpha|}{Re(\alpha)}+\frac{|\zeta-\alpha|}{Re(\zeta-\alpha)}\right)M_0^{\frac{Re(\zeta-\alpha)}{Re(\zeta)}}(t)M_1^{\frac{Re(\alpha)}{Re(\zeta)}}. \quad (13)$$

So finally by simplifying we get

$$t^\alpha\Theta^\alpha(x,y,n)\le\frac{t^{(N-1)/2}}{(\sin\gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}}+\frac{t^{(N-3)/2}}{(\sin\gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}}+$$
$$+\frac{t^{-1}}{(\sin(\gamma)/2)^{1+N}};$$

3 The estimates for maximal operators

In this section we are going to prove the estimation for maximal operators of the Riesz means by using the results of previous section. Let us recall some standard definitions from harmonic analysis on unit sphere.

Spherical ball $B(x,r)$ of radius r and with the center at point x defined by $B(x,r)=\{y\in S^N:\gamma(x,y)<r\}$. For integrable function $f(x)$ the maximal function of Hardy-Littlwood

$$f^*(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{S^N}|f(y)|d\sigma(y) \quad (14)$$

is finite almost everywhere on sphere. The maximal function f^* plays a major role in analysis and has been much studied (see.[17]). In particular, for any $p>1$ and if $f\in L_p$, then there exists constant c_p, such that

$$\|f^*\|_{L_p}\le\frac{c_p(N)}{p-1}\|f\|_{L_p},$$

where c_p has no singularities at point $p=1$.

Theorem 3.1 *Let $\alpha > \frac{N-1}{2}$ then for all $f \in L_1(S^N)$ we have*

$$E_*^\alpha f(x) \leq \frac{c_\alpha(N)}{\alpha - \frac{N-1}{2}} \left(f^*(x) + f^*(\bar{x})\right) \tag{15}$$

where $\gamma(x, \bar{x}) = \pi$.

Proof. Since Riesz means of the Fourier-Laplace series are integral operator with the kernel $\Theta^\alpha(x, y, n)$, we use the asymptotic behavior of this kernel in Theorem15.

Using this estimates for the kernel $\Theta^\alpha(x, y, n)$ we can estimate the Riesz means of the spectral expansions:

$$E_n^\alpha f(x) = \int\limits_{S^N} \Theta^\alpha(x, y, n) f(y) d\sigma(y)$$

separate into four part as follow

$$E_n^\alpha f(x) = \int\limits_{S^N} \Theta^\alpha(x, y, n) f(y) d\sigma(y) =$$

$$\int\limits_{\gamma(x,y)<\frac{1}{n}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) + + \int\limits_{\frac{1}{n}<\gamma(x,y)\leq\frac{\pi}{2}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) +$$

$$+ \int\limits_{\frac{\pi}{2}<\gamma(x,y)\leq\pi-\frac{1}{n}} \Theta^\alpha(x, y, n) f(y) d\sigma(y) + \int\limits_{\pi-\frac{1}{n}<\gamma(x,y)\leq\pi} \Theta^\alpha(x, y, n) f(y) d\sigma(y).$$

and for estimate each part let us apply the Theorem 15.

$$|E_n^\alpha f(x)| \leq C \left(n^N \int\limits_{\gamma(x,y)<\frac{1}{n}} |f(y)| d\sigma(y) + n^{\frac{N-1}{2}-\alpha} \int\limits_{\frac{1}{n}<\gamma(x,y)\leq\frac{\pi}{2}} (\sin\gamma)^{-\frac{N+1}{2}-\alpha} |f(y)| d\sigma(y) + \right.$$

$$+ n^{\frac{N-3}{2}-\alpha} \int\limits_{\frac{1}{n}<\gamma(x,y)\leq\frac{\pi}{2}} (\sin\gamma)^{-\frac{N+3}{2}-\alpha} |f(y)| d\sigma(y) + n^{-1} \int\limits_{\frac{1}{n}<\gamma(x,y)\leq\frac{\pi}{2}} (\sin\gamma)^{-1-N} |f(y)| d\sigma(y) +$$

$$+ n^N \int\limits_{\pi-\frac{1}{n}<\gamma(x,y)\leq\pi} |f(y)| d\sigma(y) + n^{\frac{N-1}{2}-\alpha} \int\limits_{\pi/2<\gamma(x,y)\leq\pi-\frac{1}{n}} (\sin\gamma)^{-\frac{N+1}{2}-\alpha} |f(y)| d\sigma(y) +$$

$$\left. + n^{\frac{N-3}{2}-\alpha} \int\limits_{\pi/2<\gamma(xy)\leq\pi-\frac{1}{n}} (\sin\gamma)^{-\frac{N+3}{2}-\alpha} |f(y)| d\sigma(y) + + n^{-1} \int\limits_{\pi/2<\gamma(x,y)\leq\pi-\frac{1}{n}} (\sin\gamma)^{-1-N} |f(y)| d\sigma(y) \right).$$

Let us denote by $U_n(x)$ and $V_n(x)$ the first four member and the last four member, respectively:

$$U_n(x) = n^N \int\limits_{\gamma(x,y)<\frac{1}{n}} |f(y)|d\sigma(y) + n^{\frac{N-1}{2}-\alpha} \int\limits_{\frac{1}{n}<\gamma(x,y)\le\frac{\pi}{2}} (\sin\gamma)^{-\frac{N+1}{2}-\alpha}|f(y)|d\sigma(y)+$$

$$+n^{\frac{N-3}{2}-\alpha} \int\limits_{\frac{1}{n}<\gamma(x,y)\le\frac{\pi}{2}} (\sin\gamma)^{-\frac{N+3}{2}-\alpha}|f(y)|d\sigma(y)+n^{-1} \int\limits_{\frac{1}{n}<\gamma(x,y)\le\frac{\pi}{2}} (\sin\gamma)^{-1-N}|f(y)|d\sigma(y)$$

and

$$V_n(x) = n^N \int\limits_{\pi-\frac{1}{n}<\gamma(x,y)\le\pi} |f(y)|d\sigma(y)+n^{\frac{N-1}{2}-\alpha} \int\limits_{\pi/2<\gamma(x,y)\le\pi-\frac{1}{n}} (\sin\gamma)^{-\frac{N+1}{2}-\alpha}|f(y)|d\sigma(y)+$$

$$+n^{\frac{N-3}{2}-\alpha} \int\limits_{\pi/2<\gamma(xy)\le\pi-\frac{1}{n}} (\sin\gamma)^{-\frac{N+3}{2}-\alpha}|f(y)|d\sigma(y)+n^{-1} \int\limits_{\pi/2<\gamma(x,y)\le\pi-\frac{1}{n}} (\sin\gamma)^{-1-N}|f(y)|d\sigma(y)$$

It is not hard to see that $U_n(\overline{x}) = V_n(x)$, where $\overline{x}$ os opposite point to $x \in S^N$, i.e. $\gamma(x,\overline{x}) = \pi$.

If we define function $F(t)$ by

$$F(t) = \int\limits_{\gamma(x,y)<t} |f(y)|d\sigma(y)$$

then it is easy to show that

$$F(t) \le C\ t^N f^*(x)$$

Due to definition of $F(t)$ we can rewrite the expression of $U_n(x)$ as follow:

$$U_n(x) = n^N F(1/n) + n^{\frac{N-1}{2}-\alpha} \int\limits_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-\frac{N+1}{2}-\alpha} F'(t)dt+$$

$$+n^{\frac{N-3}{2}-\alpha} \int\limits_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-\frac{N+3}{2}-\alpha} F'(t)dt + n^{-1} \int\limits_{\frac{1}{n}}^{\frac{\pi}{2}} (\sin t)^{-1-N} F'(t)dt.$$

Integrating by parts, we have

$$U_n(x) \le C\, f^*(x) \left(1 + n^{\frac{N-1}{2}-\alpha} \int\limits_{1/n}^{\pi/2} t^{-\frac{N+3}{2}-\alpha} dt + n^{\frac{N-3}{2}-\alpha} \int\limits_{1/n}^{\pi/2} t^{-\frac{N+5}{2}-\alpha} dt + n^{-1} \int\limits_{1/n}^{\pi/2} \frac{dt}{t^2} \right)$$

The interior of right part of the last inequality may be compute exactly and

$$U_n(x) \le \frac{C_1}{\alpha - \frac{N-1}{2}} f^*(x)$$

Analogously for $V_n(x)$ we have

$$V_n(x) \le \frac{C_2}{\alpha - \frac{N-1}{2}} f^*(\overline{x})$$

So finally for Riesz means we have
So providing the inequality $\alpha > \frac{N-1}{2}$ we have

$$E_*^\alpha f(x) \le \frac{C_\alpha(N)}{\alpha - \frac{N-1}{2}} \left(f^*(x) + f^*(\overline{x}) \right). \tag{16}$$

Theorem 3.1 is proved.

References

[1] Alimov, Sh.A., Il'in, V.A., and Nikishin, E.M.(1976/1977) *Questions of convergence of multiple trigonometric series and spectral decompositions*,Usp.Math.Nauk 31, N.6, 28083;32,N.1, 107-130. Russian Math. Surv. **31, N. 6**, (1976), 29- 86; **32, N.1**, (1977), 115-139, Zbl.345.42002; Zbl.353.42005.

[2] Alimov Sh.A., Ashurov R.R., Pulatov A.K., *Multiple Fourier Series and Fourier Integrals*Commutative Harmonic Analysis -IV, Springer-Verlag, 1992, 1-97.

[3] Ashurov, R.R., *Summability almost everywhere of fourier series in Lp with respect to eigenfunctions* (1983) Mathematical Notes of the Academy of Sciences of the USSR 34 (6), pp. 913-916

[4] Bastis, A.I., *Almost-everywhere convergence of expansions in eigenfunctions of the Laplace operator on the sphere* Mathematical Notes of the Academy of Sciences of the USSR 33 (6), pp. 440-443

[5] Bochner S, Summation of multiple Fourier series by spherical means. Trans. Amer. Math. Soc., 40 (1936), 175-207

[6] Carleson L On convergence and growth of partial sums of Fourier series, Acta. Math., 116 (1966), 135-157; MR, 33, 7774

[7] Dunford, N. and Schwartz, J.T. : Linear Operators, General Theory, and other 3 volumes, includes visualization charts

[8] Hermander L, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators, Recent advances in the Basic Sciences*, (Proc. Annual Sci. Conf. Belfer Grad. School Sci. 2, (Yeshiva Univ. New York), 1965-1966, 155-202 ; MR, 41, 2239

[9] Kogbetliantz, E. *Recherches sur la sommabilite des series ultraspheriques par la methode des moyennes arithmetiques* (1924) J. Math. Pures Appl., 9 (3), pp. 107-187.

[10] Mitchell T. *On the spherical summability of multiple orthogonal series* Trans.Amer. Math.Soc. (1951) 71, pp. 136-151

[11] Rakhimov, A.A., *On the summability of multiple trigonometric fourier series of distributions* Doklady Mathematics 62 (2), 2000, pp. 163-165

[12] Rakhimov, A.A., *Summability of the spectral expansions of distributions, connected with the elliptic partial differential operators* Thesis of Doctor Science Degree,2004, Tashkent.

[13] A.A. Akhmedov , *About the almost everywhere convergence of spectral expansions from* $L_1^{\alpha}(S^N)$, Uzbek Mathematical Journal, N. 4, 1996, pp. 17-25.

[14] A.A. Ahmedov. *About spectral expansions of Laplace - Beltrami operator on sphere*, Uzbek Mathematical Journal, N. 4, 1997, pp. 20-27.

[15] A.A. Ahmedov. *Conditions of localization averages of Riesz on spectral decomposition of the operator Laplace - Beltrami on sphere of distributions*, International Science Journal, Science. Education. Engineering., Osh, Kyrgyzstan - N.3, 2006, pp. 36-40.

[16] A.A. Ahmedov. *About convergence spectral expansions of functions from Liuovill class*, Natural and Technical science, Russia, Moscow, 2006, N.5 (25), pp.12-17.

[17] Stein E M and G Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Math. Series No. 32, (University Press, Princeton, N.J.), 1971; MR, 46 (1974), 4102;

[18] Zhizhiashvili, L.V., Topuriya, S.B. *Fourier-Laplace series on a sphere* ,Journal of Soviet Mathematics 12 (6),1979, pp. 682-714

[19] V.A. Il'in, "On a generalized interpretation of the principle of localization for Fourier series with respect to fundamental systems of functions", *Sib. Mat. Zh.,*, Vol.9, No.5, (1968), pp.1093-1106.

[20] A.J. Bastys, "The generalized localization principle for an N-fold Fourier integral", *Dokl. Akd. Nauk SSSR*, Vol.278, No.4, (1984), pp.777-778.

[21] A.J. Bastys, "Generalized localization of Fourier series with respect to the iegenfunctions of the Laplace operator in the classes L_p ", *Dokl. Akd. Nauk SSSR*, Vol.278, No.4, (1984), pp.777-778.